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Bayesian Estimation of the Parameter of New Modified Lindley Distribution Under Different Loss Functions Using Type II Censored Sample

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ABSTRACT
In this present study we have considered the Modified Lindley distribution having single parameter as a lifetime distribution. It's hazard rate function is unimodal, reverse bathtub and becomes constant for larger values of the variate value. It is true for almost all the values of its parameter θ. We have computed Bayes estimators of θ on the basis of type II censored sample from it under various such loss functions as squared error loss function(SELF), weighted squared error loss function(WSELF), modified squared error loss function(MSELF), exponentiated squared error loss function(ESELF), precautionary loss function(PLF) and logarithmic loss function(LLF). The estimators thus obtained have been compared through their posterior risks.

KEYWORDS
Bayesian estimation, Censoring scheme, Loss function, Modified Lindley distribution, Posterior risk

1. Introduction

[13] proposed the Lindley distribution having the single parameter θ, which has increasing nature of hazard rate function. It was discovered to solve the limitation of exponential distribution which has only constant shape of hazard rate function. Although, the Lindley distribution itself is restricted to model the data having increasing failure rate (IFR) only. A number of attempts has been made to generalise/transform the Lindley distribution in order to have much flexible distribution in the sense of having different shapes of hazard rate function, as compared to the Lindley distribution itself.

Recently, [6] proposed a general family of the Modified Lindley distribution by the use of a particular tuning function $w(x) = e^{-\theta x}; x > 0, \theta > 0$ and the cdf of this new modified Lindley distribution is as follows,

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$$F(x) = 1 - \left[1 + \frac{\theta x}{1+\theta} e^{-\theta x}\right] e^{-\theta x} \quad ; x > 0, \theta > 0 \tag{1}$$

The new distribution is abbreviated as ML(θ) distribution. As there is no inclusion of any additional parameter to the Lindley distribution with the parameter θ, hence it is parsimonious in parameter and consequently simple to use. Also, the pdf, survival function and hrf of ML(θ) distribution are given by,

$$f(x) = \frac{\theta}{1+\theta} e^{-2\theta x} \left[(1+\theta)e^{\theta x} + 2\theta x - 1\right] \quad ; x > 0, \theta > 0 \tag{2}$$

$$S(x) = (1+\theta)e^{\theta x} + \theta x \quad ; \forall x > 0, \theta > 0 \tag{3}$$

and

$$h(x) = \frac{\theta(\theta x - 1)}{(1+\theta)e^{\theta x} + \theta x} + \theta \quad ; \forall x > 0, \theta > 0 \tag{4}$$

respectively.

It is important to mention here that for w(x)=0, F(x) becomes the cdf of exponential(θ) distribution; while for w(x)=1, it becomes the cdf of the Lindley distribution with parameter θ. This shows that the ML(θ) distribution is somehow the generalisation of Exp(θ) distribution.

The rest of the paper is organized as follows: In section 2, we have obtained likelihood function for Type II censored sample from ML(θ) distribution. Next in section 3 we have derived the Bayes estimators of θ under SELF, WSELF, MSELF, ESELF, PLF, and LLF for the gamma prior of θ. The simulation study is carried in section 4 in order to compare the considered estimators of θ in terms of simulated risks. Finally, conclusions have been shown in the last section 5.

2. Type II Censoring

Life testing experiment refers to a procedure designed to measure the reliability characteristics of identical items by recording their "time to failure" under normal operating conditions. In life testing experiment most of the times one generally cannot observe the failure of all the items put on the test due to time and cost constraints. Therefore, adequate information and the results on failure times of all the objects put on test cannot be obtained. The data, thus obtained are called the censored data. The very first work on censored data can be found in [3] and [4]. The most commonly used censoring schemes are type I, type II, hybrid type I, hybrid type II, progressive type I, progressive type II censoring schemes etc. In the present paper, we have considered estimation of the parameters of the considered distribution on the basis of type II censored sample from it. In type II censoring, we have a total of n items put on the life testing experiment and wait for the failure of r $\leq$ n items only, where r is a pre-fixed number. The type II censored sample thus obtained is $x_{(1)} \leq x_{(2)} \leq ... \leq x_{(r)}$ and the likelihood function for this type II censored sample is given by [2].

$$L(\theta|\underline{X}) = \frac{n!}{(n-r)!}\prod_{i=1}^{r} f\left(x_{(i)},\theta\right)\left(1-F(x_{(r)},\theta)\right)^{n-r} \tag{5}$$

3. Bayes Estimators of θ

An estimator $\hat{\theta}$ is said to be Bayes estimator of θ if it minimises the Bayes risk. Let the parameter θ of ML(θ) distribution is a random variable having prior distribution as Gamma G(a,b) distribution where a is shape and b is rate parameter, which is an informative prior. Any information available for θ can be modelled by certain choice of the hyper-parameters a and b. The hyper-parameters a and b can be evaluated if we have any two independent informations available regarding them. Suppose we have known prior mean and variance as M and V respectively, so that M=$\frac{a}{b}$ and V=$\frac{a}{b^2}$ giving a=$\frac{M^2}{V}$ and b=$\frac{M}{V}$ (see [7], [8], [10], [21]). The posterior pdf of θ given Type II censored sample $\underline{X} = (X_1, X_2, ..., X_r)$ is obtained as follows:

$$h(\theta|\underline{x}) = \frac{\frac{\theta^{r+\alpha-1}}{(1+\theta)^r}e^{-\theta[\beta+2\sum_{i=1}^{r}x_i+(n-r)x_r]}\left[1+\frac{\theta x_r}{1+\theta}e^{-\theta x_r}\right]^{n-r}\prod_{i=1}^{r}\left[(1+\theta)e^{\theta x_i}+2\theta x_i-1\right]}{\int_0^\infty \frac{\theta^{r+\alpha-1}}{(1+\theta)^r}e^{-\theta[\beta+2\sum_{i=1}^{r}x_i+(n-r)x_r]}\left[1+\frac{\theta x_r}{1+\theta}e^{-\theta x_r}\right]^{n-r}\prod_{i=1}^{r}\left[(1+\theta)e^{\theta x_i}+2\theta x_i-1\right]d\theta}$$
$$= \frac{A(\theta)\phi(\theta,\underline{X})\xi(\theta,\underline{X},r)}{\int_0^\infty A(\theta)\phi(\theta,\underline{X})\xi(\theta,\underline{X},r)d\theta}$$

where $A(\theta) = \frac{\theta^{r+\alpha-1}}{(1+\theta)^r}$
$\phi(\theta,\underline{X}) = \prod_{i=1}^{r}\left[(1+\theta)e^{\theta x_i}+2\theta x_i-1\right]$
$\xi(\theta,\underline{X},r) = e^{-\theta[\beta+2\sum_{i=1}^{r}x_i+(n-r)x_r]}\left[1+\frac{\theta x_r}{1+\theta}e^{-\theta x_r}\right]^{n-r}$

3.1. *Bayes Estimators and Posterior Risks Under Different Loss Functions*

An important criterion to obtain a better Bayes estimator among the considered Bayes estimators of a parameter of interest is that it has the smallest posterior risk as compared to those of the others. This criterion is used by several authors (see [18], [19], [20], [15], [40]). Here, we have derived Bayes estimators of θ and the corresponding posterior risks under six useful loss functions, which are Squared Error Loss Function (SELF), Weighted Squared Error Loss Function (WSELF), Modified Squared Error Loss Function (MSELF), Exponentiated Squared Error Loss Function (ESELF), Precautionary Loss Function (PLF) and Logarithmic Loss Function (LLF).

3.1.1. Squared Error Loss Function (SELF)

The squared error loss function (SELF) is proposed by [42] and [43] and it is a useful loss function in the situation when over estimation and under estimation are of equal importance.

$$L_S(\theta, \hat{\theta}) = (\hat{\theta} - \theta)^2 \tag{6}$$

The Bayes estimator of θ and the corresponding posterior risk of θ are given by

$$\hat{\theta_S} = E_\theta(\theta \mid \underline{X}) \tag{7}$$

and

$$R_s(\hat{\theta}_s, \theta) = E(\theta^2 \mid \underline{X}) - [E(\theta \mid \underline{X})]^2 \tag{8}$$

and the same for the parameter θ of ML(θ) distribution for the type II censored sample $\underline{\text{X}} = (X_1, X_2, ..., X_r)$ are obtained as follows:

$$\begin{aligned} \hat{\theta}_s =& E_\theta(\theta \mid \underline{X}) = \int_0^\infty \theta h(\theta|\underline{X}) d\theta \\ =& \frac{\int_0^\infty \theta A(\theta)\phi(\theta, \underline{X})\xi(\theta, \underline{X}, r) d\theta}{\int_0^\infty A(\theta)\phi(\theta, \underline{X})\xi(\theta, \underline{X}, r) d\theta} \end{aligned} \tag{9}$$

and

$$\begin{aligned} R_s(\hat{\theta}_s, \theta) =& E(\theta^2 \mid \underline{X}) - [E(\theta \mid \underline{X})]^2 \\ =& \int_0^\infty \theta^2 h(\theta \mid \underline{x}) d\theta - \left(\int_0^\infty \theta h(\theta \mid \underline{x}) d\theta\right)^2 \\ =& \frac{\int_0^\infty \theta^2 A(\theta)\phi(\theta, \underline{X})\xi(\theta, \underline{X}, r) d\theta}{\int_0^\infty A(\theta)\phi(\theta, \underline{X})\xi(\theta, \underline{X}, r) d\theta} \\ & - \left[\frac{\int_0^\infty \theta A(\theta)\phi(\theta, \underline{X})\xi(\theta, \underline{X}, r) d\theta}{\int_0^\infty A(\theta)\phi(\theta, \underline{X})\xi(\theta, \underline{X}, r) d\theta}\right]^2 \end{aligned} \tag{10}$$

3.1.2. Weighted Squared Error Loss Function (WSELF)

The weighted squared error loss function (WSELF) is defined as

$$L_W(\hat{\theta}, \theta) = \frac{(\hat{\theta} - \theta)^2}{\theta} \tag{11}$$

Unlike SELF it has an asymmetric shape. For more details see [41]. The corresponding Bayes estimator of the parameter θ of ML(θ) distribution for type II censored sample under WSELF is given by

$$\begin{aligned}
\hat{\theta_w} &= \left[E_\theta(\theta^{-1} \mid \underline{X})\right]^{-1} \\
&= \left[\int_0^\infty \theta^{-1} h(\theta \mid \underline{X})\right]^{-1} \\
&= \left[\frac{\int_0^\infty \theta^{-1} A(\theta)\phi(\theta, \underline{X})\xi(\theta, \underline{X}, r)d\theta}{\int_0^\infty A(\theta)\phi(\theta, \underline{X})\xi(\theta, \underline{X}, r)d\theta}\right]^{-1}
\end{aligned} \tag{12}$$

and the posterior risk of θ_w is :

$$\begin{aligned}
R_w(\hat{\theta}, \theta) =& E_\theta\left[\theta \mid \underline{X}\right] - \left[E_\theta\left[\theta^{-1} \mid \underline{X}\right]\right]^{-1} \\
=& \frac{\int_0^\infty \theta A(\theta)\phi(\theta, \underline{X})\xi(\theta, \underline{X}, r)d\theta}{\int_0^\infty A(\theta)\phi(\theta, \underline{X})\xi(\theta, \underline{X}, r)d\theta} \\
&- \left[\frac{\int_0^\infty \theta^{-1} A(\theta)\phi(\theta, \underline{X})\xi(\theta, \underline{X}, r)d\theta}{\int_0^\infty A(\theta)\phi(\theta, \underline{X})\xi(\theta, \underline{X}, r)d\theta}\right]^{-1}
\end{aligned} \tag{13}$$

3.1.3. Modified Squared Error Loss Function (MSELF)

[23] proposed the modified squared error loss function (MSELF) defined as:

$$L_M(\theta, \theta) = \left[\frac{\hat{\theta} - \theta}{\theta}\right]^2 \tag{14}$$

Like SELF it is also symmetric in shape and the corresponding Bayes estimator of the parameter θ of ML(θ) distribution for type II censored sample under MSELF is given by

$$\begin{aligned}
\hat{\theta_M} =& \frac{E_\theta\left[\theta^{-1} \mid \underline{X}\right]}{E_\theta\left[\theta^{-2} \mid \underline{X}\right]} \\
=& \frac{\left[\int_0^\infty \theta^{-1} h(\theta \mid \underline{X})\right]}{\left[\int_0^\infty \theta^{-2} h(\theta \mid \underline{X})\right]} \\
=& \frac{\int_0^\infty \theta^{-1} A(\theta)\phi(\theta, \underline{X})\xi(\theta, \underline{X}, r)d\theta}{\int_0^\infty \theta^{-2} A(\theta)\phi(\theta, \underline{X})\xi(\theta, \underline{X}, r)d\theta}
\end{aligned} \tag{15}$$

and the posterior risk of $\hat{\theta_M}$ is:

$$
\begin{aligned}
R_M(\hat{\theta},\theta) =& 1-\frac{\left[E_\theta\left[\theta^{-1}\ |\underline{X}\right]\right]^2}{E_\theta\left[\theta^{-2}\ |\underline{X}\right]}\\
=& 1-\frac{\left[\left[\int_0^\infty \theta^{-1}h(\theta\ |\underline{X})\right]\right]^2}{\left[\int_0^\infty \theta^{-2}h(\theta\ |\underline{X})\right]}\\
=& 1-\frac{\left[\int_0^\infty \theta^{-1}A(\theta)\phi(\theta,\underline{X})\xi(\theta,\underline{X},r)d\theta\right]^2}{\int_0^\infty \theta^{-2}A(\theta)\phi(\theta,\underline{X})\xi(\theta,\underline{X},r)d\theta\int_0^\infty A(\theta)\phi(\theta,\underline{X})\xi(\theta,\underline{X},r)d\theta}
\end{aligned}
\tag{16}
$$

3.1.4. Precautionary loss function (PLF)

Traditionally, Bayes parameter estimation is based on a quadratic loss-function but [14] introduced an alternative non-symmetric Precautionary loss function which is defined as:

$$
L_P(\hat{\theta},\theta) = \frac{\left[\hat{\theta}-\theta\right]^2}{\hat{\theta}} \tag{17}
$$

The Bayes estimator of the parameter θ of ML(θ) distribution under PLF is given by:

$$
\begin{aligned}
\hat{\theta_P} =& \sqrt{E_\theta(\theta^2\ |\underline{X})}\\
=& \sqrt{\int_0^\infty \theta^2 h(\theta|\underline{X})d\theta}\\
=& \sqrt{\frac{\int_0^\infty \theta^2 A(\theta)\phi(\theta,\underline{X})\xi(\theta,\underline{X},r)d\theta}{\int_0^\infty A(\theta)\phi(\theta,\underline{X})\xi(\theta,\underline{X},r)d\theta}}
\end{aligned}
\tag{18}
$$

and the corresponding posterior risk of the Bayes estimator $\hat{\theta_P}$ of the parameter is:

$$
\begin{aligned}
R_P(\hat{\theta_P},\theta) =& 2\left[\sqrt{E_\theta(\theta^2\ |\underline{X})}-E_\theta(\theta\ |\underline{X})\right]\\
=& 2\left[\sqrt{\int_0^\infty \theta^2 h(\theta|\underline{X})d\theta}-\int_0^\infty \theta h(\theta|\underline{X})d\theta\right]\\
=& 2\left[\sqrt{\frac{\int_0^\infty \theta^2 A(\theta)\phi(\theta,\underline{X})\xi(\theta,\underline{X},r)d\theta}{\int_0^\infty A(\theta)\phi(\theta,\underline{X})\xi(\theta,\underline{X},r)d\theta}}-\frac{\int_0^\infty \theta A(\theta)\phi(\theta,\underline{X})\xi(\theta,\underline{X},r)d\theta}{\int_0^\infty A(\theta)\phi(\theta,\underline{X})\xi(\theta,\underline{X},r)d\theta}\right]
\end{aligned}
\tag{19}
$$

3.1.5. Logarithmic Loss Function (LLF)

The Logarithmic loss function is defined as:

$$L_L(\hat{\theta},\theta) = \left[\ln\hat{\theta} - \ln\theta\right]^2 \tag{20}$$

The Bayes estimator of the parameter θ of ML(θ) distribution under Logarithmic loss function is given by:

$$\begin{aligned}\hat{\theta_L} =& e^{E(\ln\theta\ |\underline{X})}\\ =& e^{\int_0^\infty \ln\theta h(\theta\ |\underline{X})d\theta}\\ =& e^{\left[\frac{\int_0^\infty \ln\theta A(\theta)\phi(\theta,\underline{X})\xi(\theta,\underline{X},r)d\theta}{\int_0^\infty A(\theta)\phi(\theta,\underline{X})\xi(\theta,\underline{X},r)d\theta}\right]}\end{aligned} \tag{21}$$

and the corresponding posterior risk is:

$$\begin{aligned}R_L(\hat{\theta},\theta) =& Var(\ln\theta)\\ =& \int_0^\infty (\ln\theta)^2\, h(\theta|\underline{X})d\theta - \left[\int_0^\infty (\ln\theta)\, h(\theta|\underline{X})d\theta\right]^2\\ =& \left[\frac{\int_0^\infty (\ln\theta)^2\, A(\theta)\phi(\theta,\underline{X})\xi(\theta,\underline{X},r)d\theta}{\int_0^\infty A(\theta)\phi(\theta,\underline{X})\xi(\theta,\underline{X},r)d\theta}\right]\\ &- \left[\frac{\int_0^\infty (\ln\theta)\, A(\theta)\phi(\theta,\underline{X})\xi(\theta,\underline{X},r)d\theta}{\int_0^\infty A(\theta)\phi(\theta,\underline{X})\xi(\theta,\underline{X},r)d\theta}\right]^2\end{aligned} \tag{22}$$

3.1.6. Exponentiated SELF (ESELF)

[10] introduced a new asymmetric loss function Exponentiated SELF (ESELF), which is defined as:

$$L_E(\hat{\theta},\theta) = \left[e^{-\hat{\theta}} - e^{-\theta}\right]^2 \tag{23}$$

The Bayes estimator of the parameter θ is given by:

$$\begin{aligned}\theta_E =& -\ln\left[E(e^{-\theta}\ |\underline{X})\right]\\ =& -\ln\left[\int_0^\infty e^{-\theta}h(\theta|\underline{X})d\theta\right]\\ =& -\ln\left[\frac{\int_0^\infty e^{-\theta}A(\theta)\phi(\theta,\underline{X})\xi(\theta,\underline{X},r)d\theta}{\int_0^\infty A(\theta)\phi(\theta,\underline{X})\xi(\theta,\underline{X},r)d\theta}\right]\end{aligned} \tag{24}$$

and the corresponding posterior risk is:

$$
\begin{aligned}
R_E(\hat{\theta_E}, \theta) =& Var\left[E\left[e^{-\theta}\ |\underline{X}\right]\right] \\
=& \int_0^\infty e^{-2\theta} h(\theta|\underline{X}) d\theta - \left[\int_0^\infty e^{-\theta} h(\theta|\underline{X}) d\theta\right]^2 \\
=& \left[\frac{\int_0^\infty e^{-2\theta} A(\theta)\phi(\theta,\underline{X})\xi(\theta,\underline{X},r)d\theta}{\int_0^\infty A(\theta)\phi(\theta,\underline{X})\xi(\theta,\underline{X},r)d\theta}\right] \\
&- \left[\frac{\int_0^\infty e^{-\theta} A(\theta)\phi(\theta,\underline{X})\xi(\theta,\underline{X},r)d\theta}{\int_0^\infty A(\theta)\phi(\theta,\underline{X})\xi(\theta,\underline{X},r)d\theta}\right]^2
\end{aligned}
\tag{25}
$$

4. Simulation Study

In this section, simulation study is carried out to compare the considered Bayes estimators ($\hat{\theta}_S$, $\hat{\theta}_W$, $\hat{\theta}_M$, $\hat{\theta}_E$, $\hat{\theta}_P$ and $\hat{\theta}_L$) of the parameter θ of $ML(\theta)$- distribution in terms of their posterior risks. The Bayes estimator of the parameter θ with the smallest posterior risk is considered as the best among all the considered Bayes estimators of the parameter θ. For this study we have chosen arbitrarily $n = 15, 20, 30$ and 60 ; $r = 40\%, 60\%$ and 80% of the value of n and $\theta = 2.5, 3.0$ and 3.5. In order to get the convergence of the results, we have simulated 2000 different samples as per the plan discussed above and the results are summarised in Tables 1-5. On comparing the results of Table 1 with the corresponding results of Tables 2 and 3, we observed that in almost all the cases the posterior risk of $\hat{\theta_S}$ is least in the case when M = θ as compared to those of $\hat{\theta_S}$ in the cases when M < θ and M > θ. While, the other considered estimators $\hat{\theta_W}$, $\hat{\theta_M}$, $\hat{\theta_P}$, $\hat{\theta_L}$ and $\hat{\theta_E}$ having lowest posterior risks in almost all the cases in the case when M > θ as compared to those of the corresponding estimators in the cases when M = θ and M < θ. In addition to these findings, it is also observed that as V increases for fixed values of n and r, the posterior risks of each of the considered estimators also increases, which is a desired characteristic of the estimator that the posterior risks increases with increase in prior variance. From these tables, it is also found that for all the considered values of V, the posterior risks of various considered estimators of θ decreases with increase in the considered value of r (which is 40%, 60%, and 80% of the considered value of n = 30), which is also a desired result.

Further, Tables 4 and 5 provides the simulated posterior risks of the above considered estimators of θ for the different values of r = 40%, 60% and 80% of the various values of n = 15, 20, 30 and 60 and the various values of θ are taken as 2.5 (Table 4) and 3.5 (Table 5). The values of M and V are taken as 3 and 1 resp. From these tables, it is observed that the posterior risks of the various considered estimators of θ decreases as r increases for fixed n. It is true for every considered fixed values of n. In addition, for a fixed percentage of different value of n (effective sample size r), the simulated posterior risks of the considered estimators of θ decreases with increase in

Table 1. Posterior Risks of the estimator of θ when prior variance varies for fixed $n = 30$, $r(r = 12, 18, 24)$, $\theta = 3.0$, $M = 2.5$

V	scheme r	$R(\hat{\theta})_S$	$R(\hat{\theta})_{WS}$	$R(\hat{\theta})_{MS}$	$R(\hat{\theta})_{PLF}$	$R(\hat{\theta})_{LLF}$	$R(\hat{\theta})_{ES}$
0.5	12	0.2500142	0.09289302	0.03575057	0.08606901	0.03269905	0.001156316
	18	0.2397845	0.06944081	0.02137417	0.07533632	0.02511547	0.00042651
	24	0.2389352	0.05524527	0.01729827	0.06912821	0.02012459	0.000319675
1	12	0.3588064	0.1219937	0.04356605	0.1191209	0.0417142	0.001270737
	18	0.3423804	0.1090543	0.03063508	0.1098424	0.03573746	0.00050226
	24	0.3159132	0.1086892	0.02987128	0.1049506	0.02950652	0.000311668
2	12	0.4486023	0.145864	0.05001015	0.1444736	0.04865429	0.001349253
	18	0.4037468	0.1257692	0.03587759	0.1281272	0.03652177	0.000547435
	24	0.3920611	0.1160097	0.03107977	0.1109537	0.0300483	0.000375311
5	12	0.522845	0.1650825	0.05512198	0.1643243	0.05396075	0.001400811
	18	0.4456042	0.1400075	0.03920596	0.141064	0.03947367	0.000573064
	24	0.4147355	0.1239611	0.0360842	0.1212924	0.03608319	0.000377618
80	12	0.5814825	0.1798207	0.05895879	0.1794048	0.05787395	0.001433154
	18	0.5637428	0.1502728	0.04342856	0.1507892	0.04347574	0.000604162
	24	0.5015473	0.1312255	0.03701516	0.1297143	0.03916693	0.000429041

Table 2. Poterior Risks of the estimator of θ when prior variance varies for fixed $n = 30$, $r(r = 12, 18, 24)$, $\theta = 3.0$, $M = 3.0$

V	scheme r	$R(\hat{\theta})_S$	$R(\hat{\theta})_{WS}$	$R(\hat{\theta})_{MS}$	$R(\hat{\theta})_{PLF}$	$R(\hat{\theta})_{LLF}$	$R(\hat{\theta})_{ES}$
0.5	12	0.2261726	0.07115002	0.02416211	0.07293669	0.02324175	0.00057422
	18	0.2050894	0.06885996	0.02180294	0.06816154	0.02033037	0.000421845
	24	0.1947531	0.06120579	0.0192722	0.06273738	0.01675707	0.000367823
1	12	0.3520407	0.1115223	0.03751239	0.1116995	0.03626897	0.000904453
	18	0.3260565	0.106634	0.02985623	0.0903247	0.03124571	0.000446311
	24	0.2944187	0.0924279	0.02531779	0.07638943	0.0270732	0.000391145
2	12	0.4455123	0.1402123	0.04664217	0.1397079	0.04541752	0.00112957
	18	0.4103479	0.1245269	0.03489725	0.126055	0.03554295	0.000511707
	24	0.3679894	0.1105541	0.02979251	0.0971922	0.03052794	0.000450927
5	12	0.5214153	0.1625391	0.05359054	0.1619605	0.05244515	0.001298962
	18	0.5139579	0.1389009	0.03862642	0.1396346	0.03887607	0.000587415
	24	0.4589482	0.1219938	0.03048243	0.1193643	0.03344046	0.000477484
80	12	0.5813751	0.1796431	0.06185167	0.1792315	0.05776678	0.001426041
	18	0.5235649	0.1501768	0.04438392	0.1510679	0.04942968	0.000712427
	24	0.4838571	0.1327316	0.03732544	0.1268523	0.03870723	0.000609983

the value of n. And last but not the least ESELF ($\hat{\theta}_{ES}$) can be considered as the best estimator in terms of lowest posterior risk, for all the considered criteria i.e., varying prior mean and prior variance keeping θ fixed (Tables 1, 2 & 3) and varying θ while keeping prior mean and variance fixed.

5. Conclusion

In the present study, we have considered the new Modified Lindley ML(θ) distribution as a lifetime distribution and carried out the estimation of the parameter θ in the Bayesian paradigm by considering a type-II censored sample from it which has remained unexplored before. Here six loss functions viz. SELF, WSELF, MSELF, PLF, LLF, ESELF have been taken into consideration and the corresponding Bayes estimators are obtained for each of the six loss functions and for comparing these estimators their posterior risks have also been obtained. Firstly, in the cases when V and r increases for the fixed values of n, θ and M (see, Tables 1-3), it is observed that the

Table 3. Posterior Risks of the estimator of θ when prior variance varies for fixed $n = 30$, $r(r = 12, 18, 24)$, $\theta = 3.0$, $M = 3.5$

V	scheme r	$R(\hat{\theta})_S$	$R(\hat{\theta})_{WS}$	$R(\hat{\theta})_{MS}$	$R(\hat{\theta})_{PLF}$	$R(\hat{\theta})_{LLF}$	$R(\hat{\theta})_{ES}$
	12	0.2993414	0.08153518	0.02249766	0.08946226	0.02485334	0.000339106
0.5	18	0.2676384	0.07754429	0.01947566	0.06958786	0.0204429	0.000306303
	24	0.2138618	0.07422397	0.01331966	0.06129614	0.0169788	0.000207985
	12	0.3709009	0.107559	0.03264226	0.1114024	0.03308645	0.000598369
1	18	0.3466758	0.1001836	0.02504837	0.1093883	0.030596	0.000349364
	24	0.2840969	0.09044485	0.02199408	0.08377309	0.0230448	0.000281437
	12	0.45072	0.1356058	0.04305566	0.1365403	0.042344	0.000896117
2	18	0.4061585	0.1037817	0.03410569	0.1135728	0.03445551	0.000403259
	24	0.3731882	0.0978299	0.02797876	0.1017768	0.02655761	0.000217633
	12	0.5232704	0.1601765	0.05185495	0.159926	0.05078767	0.001176638
5	18	0.4842229	0.1380362	0.03803883	0.1382412	0.03821349	0.000455955
	24	0.4142021	0.1198012	0.03072528	0.1173815	0.02966106	0.000183277
	12	0.5815184	0.1794776	0.05872744	0.1770736	0.05764292	0.001416754
80	18	0.533561	0.1500975	0.04133479	0.1505692	0.04137764	0.000500078
	24	0.4709094	0.1309604	0.0359257	0.1294693	0.03207637	0.000267481

Table 4. Posterior Risks of the estimators of θ when $V = 1$, $M = 3$and for true value of $\theta = 2.5$

n	scheme r	$R(\hat{\theta})_S$	$R(\hat{\theta})_{WS}$	$R(\hat{\theta})_{MS}$	$R(\hat{\theta})_{PLF}$	$R(\hat{\theta})_{LLF}$	$R(\hat{\theta})_{ES}$
	6	0.3606857	0.1397295	0.05738046	0.138923	0.05605552	0.002629615
15	9	0.2976955	0.1140445	0.04604612	0.1345447	0.04534513	0.00202841
	12	0.2445852	0.1105338	0.04312573	0.1310218	0.04021528	0.001353883
	8	0.3358263	0.1043574	0.0553074	0.09347166	0.05480003	0.00241897
20	12	0.3020094	0.1016906	0.04279344	0.06034711	0.03860321	0.001321009
	16	0.2537699	0.06242372	0.04267603	0.05993998	0.02831083	0.001018886
	12	0.298234	0.07786202	0.03007343	0.08021385	0.0277044	0.000976123
30	18	0.28151166	0.06647112	0.02399845	0.01925851	0.02578795	0.000483908
	24	0.22032864	0.04264132	0.02167798	0.01462106	0.02513484	0.000451107
	24	0.2511233	0.0742954	0.0247088	0.04083595	0.02246728	0.000325126
60	36	0.2667663	0.06373139	0.01997158	0.01522118	0.01800598	0.000105621
	48	0.2056744	0.03782089	0.015182073	0.0117755	0.01559204	0.00010123

Table 5. Posterior Risks of the estimator of θ when $V = 1$, $M = 3$ and for true value of $\theta = 3.5$

n	scheme r	$R(\hat{\theta})_S$	$R(\hat{\theta})_{WS}$	$R(\hat{\theta})_{MS}$	$R(\hat{\theta})_{PLF}$	$R(\hat{\theta})_{LLF}$	$R(\hat{\theta})_{ES}$
	6	0.4692241	0.1606578	0.05819343	0.1596289	0.05698069	0.001906615
15	9	0.4113073	0.1363811	0.04721079	0.1357183	0.04658155	0.001341456
	12	0.3992768	0.1292878	0.04059289	0.1287028	0.03991055	0.000800744
	8	0.4245722	0.1121099	0.04875041	0.1520251	0.04932818	0.001761632
20	12	0.4093351	0.0983797	0.03930501	0.09562255	0.03721292	0.001142828
	16	0.3604881	0.09249124	0.03072212	0.09176044	0.03152316	0.000804193
	12	0.387382	0.1107188	0.03950127	0.1439654	0.03928317	0.00170465
30	18	0.3420292	0.088403	0.02996385	0.079565	0.02910358	0.00031574
	24	0.3139588	0.0781563	0.02449849	0.060641	0.02512219	0.000139359
	24	0.3645326	0.1072355	0.03273338	0.1351052	0.02789239	0.001169238
60	36	0.08217661	0.01483188	0.02812226	0.02065503	0.003696297	0.000264753
	48	0.0345631	0.01219155	0.01387634	0.01028559	0.01873107	0.000109886

posterior risk of $\hat{\theta_S}$, for M = θ is least as compared to that of $\hat{\theta_S}$ for M $<\theta$ and M $>\theta$; while the posterior risk of the other estimators $\hat{\theta_W}$, $\hat{\theta_M}$, $\hat{\theta_P}$, $\hat{\theta_L}$ and $\hat{\theta_E}$ for M $>\theta$ is found least as compared to the respective posterior risks of these five estimators for M $=\theta$ and M $<\theta$ in almost all the cases.

On the other hand, when n and r increases for fixed values of V and M and for fixed θ either less than M or greater than M, it is observed that the posterior risks of the various considered Bayes estimators of θ decreases in almost all the cases as it is the expected result because of the increase in the information. It is also observed that the posterior risks of the various Bayes estimators of θ for M $<\theta$ is least as compared to that of respective values for M $>\theta$. An interesting result which is to be noted is that $\hat{\theta_E}$ outperforms other estimators $\hat{\theta_S}$, $\hat{\theta_w}$, $\hat{\theta_M}$, $\hat{\theta_P}$, $\hat{\theta_L}$ in terms of having smallest posterior risk for almost all the considered values of n, r, M, V and θ.

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